Statistical indicators as potential early signals of transitions in time series obtained by a statistical model: geomagnetic field case

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Abstract

We have used a statistical model known as the “domino model” to simulate long time series of dipolar geomagnetic field. The simulated dipolar field time series, like the observed palaeomagnetic time series, has two stable states. The magnitude of the dipolar field in both cases oscillates irregularly between these states and these transitions, known as reversals of the dipolar geomagnetic field, seemingly occur at random. However there can be possible statistical indicators that serve as early signals about the incoming transitions. We applied some statistical tools like standard deviation, variance and power spectrum density (PSD) estimation to the simulated time series. All the statistical tools we have used show substantial change when approaching the eminent transition. Large values of the window width for the calculation of the standard deviation and variance indicate about the transition at earlier times. Also the PSD is affected when approaching the transition because the contribution to low frequencies is increased. Therefore we can identify these statistical tools as early indicators of future potential transitions.

Key Words: statistical model, time series, geomagnetic field, dipolar geomagnetic reversal, numerical simulation

Introduction

Many complex systems exhibit sharp transitions that occur without any noticeable warning sign (Dakos et al., 2012). Such transitions have been observed in climate (Lenton et al., 2008), lakes and coral reefs (Scheffer et al., 2001), financial markets (Johannes, 2004) or ecological systems (Dakos et al., 2012). Often in these systems can be one or more thresholds points where, depending on the respective dynamics there can be transitions to different states or regimes (Strogatz, 1994). Of special interest is the critical transition during which a given system jumps from one state to another rather distant one. This transition is reflected in drastic changes in real world systems.

Often these transitions are not predictable. However there seems to be some indicators that somehow anticipate the transition (Dakos et al., 2012). There have been extensive studies to find such indicators and many are applied to different real-world systems. In table 1 of Dakos et al., 2012 are reported many results (together with the respective references) for many indicators starting from statistical parameters like standard deviation, variance, skewness, kurtosis, autocorrelation etc., to more sophisticated models like potential well estimator or autoregressive model of order *p*.

Ideally these indicators should be tested on real world data time series. But when not available, there are used synthetic data produced by numerical models that somehow mimic the real systems. This is the case for the time series of the geomagnetic field that cannot be long enough to permit the identification of any possible indicator. Therefore, those series are generated by various numerical dynamo models. They have an expressed random nature (Shmitt et al., 2001) and physically this randomness is due to the complicated processes that occur in the liquid outer core of the Earth. The geomagnetic field has one main dipolar field ingredient (Jacobs, 1994) and minor contributions from higher order moments of the geomagnetic field that are more complicated in nature. In the long history of the geomagnetic field there have occurred drastic transitions of the dipolar field from one polarity to the other. To visualize the polarity of the geomagnetic field, take the opposite direction of a magnetic needle. These changes in polarity are known as reversals and happen in an irregular fashion indicating their random nature. In this paper we will use the term phase transition for the

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reversals. The two phases are respectively the normal polarity state and reversed polarity state. By normal polarity we mean the current polarity of the geomagnetic field. The identification of any tool would possibly enable us to forecast these events. Also it would be very helpful in understanding the underlying phenomena that govern the time evolution of the geomagnetic field.

The time series produced by the complex dynamo models are very expensive to compute due to the complexity of the equations employed. The use of a class of simpler models, often referred to as statistical models or toy models, generally eliminate the computational obstacles. Although there may be a loss in the reproducibility of the details of the real geomagnetic field, the main features are preserved. The "domino" model is one of these models, and is studied extensively (Mori et al., 2013; Duka et al., 2015; Peqini et al., 2015). The results of these authors show that this model is adequate to describe not only the reversals but also other phenomena like the secular variation (SV) or rate of change of the geomagnetic field. In this paper we analyse the time series produced by the "domino" model from the perspective of statistical analysis and are interested in identifying possible early warning signals for the phase transitions that in the case of the geomagnetic field are the reversals.

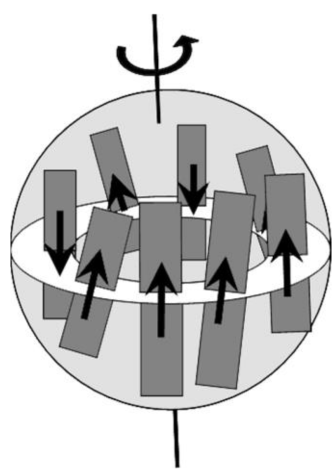
The structure of the paper is as follows: in the next section is shortly described the "domino" model. In the following section are introduced the statistical tools we use in this study and is explained the method that is applied for each of them. Then the results are reported in the next section followed by the discussions and conclusions and some recommendations for further studies in the last section.

Model

The model we study in this paper is known as the "domino" model. It inspired from several physical assumptions that are embodied in weakly driven dynamos (Davidson, 2013). The dynamo mechanism takes place in the liquid outer core of the Earth where in the weakly driven dynamo regime is organised in so called columnar convection cells. These structures are identified in numerical simulations (Kageyama and Sato, 1997). Further details on the subject and in the underlying physical details can be found in the papers Duka et al., 2015, Peqini et al., 2015 and the references therein.

The "domino" model consists of a circular arrangement of *N* macro–spins that interact pair–wise. These macro–spins are embedded in a uniformly rotating medium with unit angular velocity $\mathbf{\Omega} = (0,1)$ along the rotational axis (fig. 1). Each of the macro–spins has unit length and they can be described dynamically through the angle each of them forms with the rotational axis. Consequently an individual macro–spin is $\mathbf{S}_i = (\sin\theta_i, \cos\theta_i)$.

Figure 1: Sketch of the "domino" model.



Two essential assumptions of the model are: the tendency of each macro–spin to align with the rotation axis (dynamics in systems of rotating fluids) and the spin–spin interactions of the macro–spins. In the latter case is adopted an Ising–like interaction. Mathematically these interactions compose the potential energy which reads

$$P(t) = \gamma \sum_{i=1}^{N} \left(\mathbf{\Omega} \cdot \mathbf{S}_i \right)^2 + \lambda \sum_{i=1}^{N} \left(\mathbf{S}_i \cdot \mathbf{S}_{i+1} \right), \tag{1}$$

Where γ and λ characterise numerically the alignment tendency and the spin–spin interaction respectively. Also when $i = N$, $\mathbf{S}_{i+1} = \mathbf{S}_i$. The second term in equation (1) actually assumes an interaction among neighbouring macro–spins. Other fashions may be used like the long range coupling "domino" model (Duka et al., 2015; Peqini et al., 2015) but in this paper we focus in this type of interaction only. The kinetic energy of the system of macro–spins is

$$K(t) = \frac{1}{2} \sum_{i=1}^{N} \dot{\theta}_i^2. \tag{2}$$

A Lagrangian $L = K(t) - P(t)$ is written, where more explicitly we write

$$L = \frac{1}{2} \sum_{i=1}^{N} \dot{\theta}_i^2 - \gamma \sum_{i=1}^{N} \left(\mathbf{\Omega} \cdot \mathbf{S}_i \right)^2 - \lambda \sum_{i=1}^{N} \left(\mathbf{S}_i \cdot \mathbf{S}_{i+1} \right). \tag{3}$$

Then a Langevin – type equation is set up as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i} \right) = \frac{\partial L}{\partial \theta_i} - \kappa \dot{\theta}_i + \frac{\varepsilon \chi_i}{\sqrt{\tau}}, \tag{4}$$

where the term $\kappa \dot{\theta}_i$ describes energy dissipation and κ is the parameter of dissipation; $\varepsilon \chi_i / \sqrt{\tau}$ is the random force acting on each spin whose strength is characterised by the parameter ε. χ_i is a Gaussian-distributed random number with zero mean and unit variance and is associated to each macro-spin. The value of this random variable is updated each correlation time τ. Substituting 3 into 4 yields the system of differential equations of the model

$$\ddot{\theta}_i - 2\gamma \cos\theta_i \sin\theta_i + \lambda \left[\cos\theta_i \left(\sin\theta_{i-1} + \sin\theta_{i+1} \right) - \sin\theta_i \left(\cos\theta_{i-1} + \cos\theta_{i+1} \right) \right] + \kappa \dot{\theta}_i - \frac{\varepsilon \chi_i}{\sqrt{\tau}} = 0. \tag{5}$$

The model has a system of second order ordinary differential equations (ODE) and periodic boundary conditions are employed, where i = 1, 2, ... , N, $\theta_0 = \theta_N, \theta_{N+1} = \theta_1$.

The system of N second order ODEs is transformed into a system of $2N$ first order ODEs by making standard substitutions (Duka et al., 2015; Peqini et al., 2015). Then we integrated them with a 4th order Runge – Kutta algorithm in the MATLAB platform using an internal function (ode45). The initial values of the angles θ_i are random and uniformly distributed in (0, 2π), whilst the macro–spins are considered to be initially at rest. In summary, the "domino" model has 6 independent parameters: N, γ, λ, κ, ε and τ. With except of the last parameter which is a technical detail of the numerical simulations, the remaining parameters characterise numerically the physical processes modelled with the respective terms.

The output of each simulation is the cumulative orientation of all macro – spins or also known as axial orientation. We will refer to it as magnetisation and actually it is the scaled non – dimensional quantity corresponding to the dipolar geomagnetic moment. It is calculated by

$$M = \frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{\Omega}\cdot\mathbf{S}_i\right) = \frac{1}{N}\sum_{i=1}^{N}\cos\theta_i\left(t\right). \tag{6}$$

Statistical parameters

The standard deviation and variance are two of the statistical parameters we analyse in the present study and are shown to give positive results in forecasting transitions in time series (Carpenter and Brock, 2006). The former parameter gives an estimate of the degree by which the members of a group differ from the mean values of the actual group. The latter parameter gives a measure of how far apart are a set of (random) numbers from their mean. From these definitions one can conclude that these parameters are very similar. Hence there are expected similar results when applied to time series.

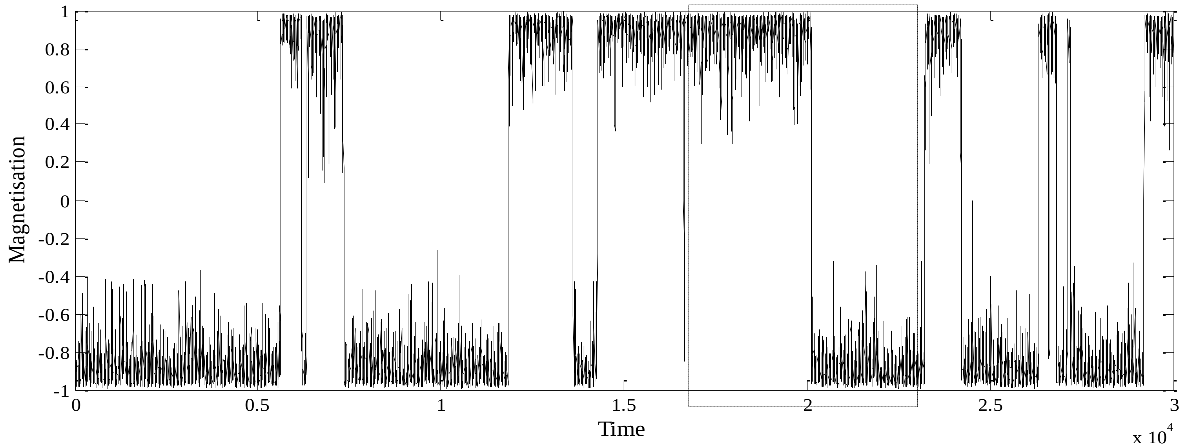
Each of the statistical parameters is calculated for moving windows of different widths. There are chosen several values that span from 17 to 1000 of model's time units. The smallest window width (in terms of the model time scale) corresponds to the smallest recorded interval between two consecutive reversals in the history of the Earth that is approximately 20,000 years (Gubbins, 1999; Duka et al., 2015). The standard deviation and variance are calculated for each window and there are obtained time series of these quantities. These time series are plotted in the same frame with the magnetisation time series for comparison.

The last statistical parameter is based on the calculation of the power spectral density (PSD) of portions of the time series. In principle a time series can be decomposed in a unique Fourier series with individual terms for specific frequencies. The PSD shows the amount of energy per unit of time that is comprised in individual frequencies. Rare or low frequency phenomena have small PSD values and high frequency phenomena have large PSD values. Phase transitions are very rare compared to other temporal variation phenomena and their presence is reflected with an increase in the low frequency edge of PSD plot. The PSD results to be successful in forecasting transitions in many time series (Kleinen et al., 2003) and seems obvious to apply it to the time series generated by the domino model. The PSD is calculated for a standard window with width of 1000 time units. This optimal value is found empirically. The window is then moved by a step of 500 time units and several PSDs are obtained.

Results

In fig. 2 is shown a typical time series generated by the "domino model". The full run comprises 300,000 time units, while in the figure are shown the first 30,000 time units. The random nature of the results of the model is evident. Also it is clear the existence of two symmetric stable states and that the magnitude of the magnetisation oscillates irregularly between them. Physically the symmetry arises from the symmetry in the dynamical equations in the liquid outer core (Rüdiger and Hollerbach, 2004). In support of this picture is also a statistical model of the amplitude of the dipolar moment known or bistable geodynamo model (Schmitt et al., 2001).

Figure 2: Magnetisation for a typical run where are shown the first 30,000 time units. Inside the dashed box is the portion studied in this paper



The effect of the values of the coefficients is substantial to the dynamics of the magnetisation and consequently to the time series produced by the model. The parameters space of the "domino" model is studied by several authors (Mori et al., 2013; Duka et al., 2015; Peqini et al., 2015) although it requires additional work to have a more complete view of different regimes. However we will consider typical parameters' values (Peqini et al., 2015): $N = 8$, $\gamma = -1$, $\lambda = -2$, $\kappa = 0.1$, $\varepsilon = 0.4$ and $\tau = 0.01$ (equal to the time step). Actually we show in fig. 1 a hundredth of the full time steps such that the time series would be easier to study. The fact that the correlation time is equal to the time step, this means that the Gaussian distributed number is updated after each time step.

In our study we focus on one piece of the full time series, from 17,000-23,000 time units, which is inside the dashed box shown in fig. 2. The fact that the statistical nature of the time series does not change in all its extension guaranties that the analysis with the statistical parameters is independent on where on the series we do apply these parameters. From this point of view there is no particular reason why we chose the specific piece of the time series. In this portion of the series is only one reversal, i.e. one phase transition from the normal polarity to the reversed polarity. Also there are present some minor variations that indicate of failed attempts to reverse polarity.

Figure 3: Standard deviation (dashed) and variance (dotted) for the interval with width 17 time units.

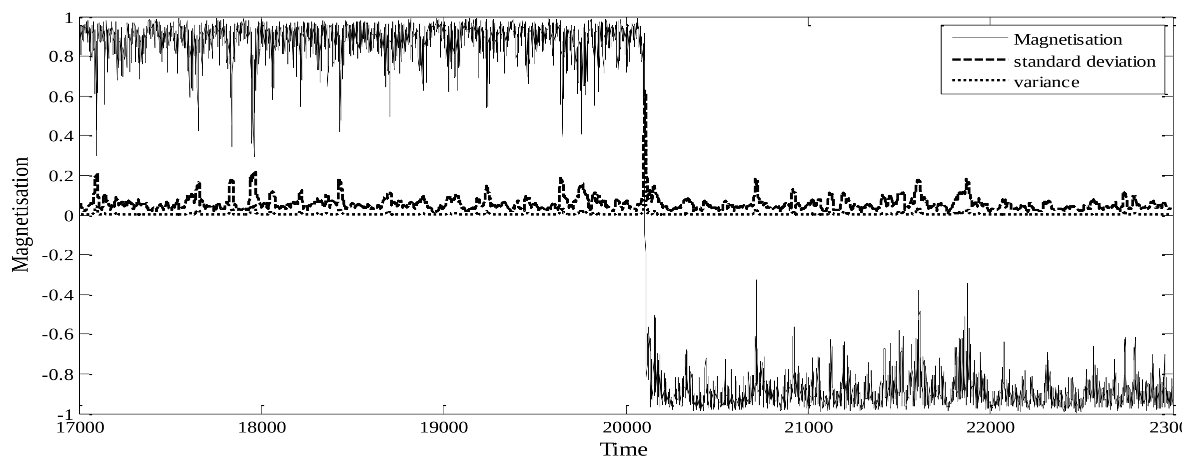
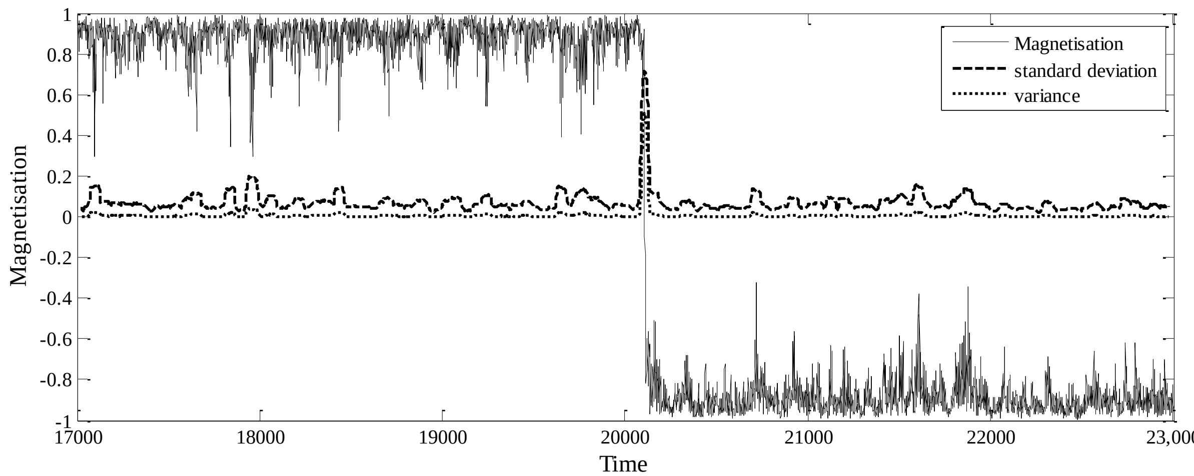


Figure 4: Standard deviation (dashed) and variance (dotted) for the interval with width 50 time units.



The first statistical parameters we study are standard deviation and variance. In the figure 3-6, the window width employed is different. The minimal width is 17 time units. This corresponds to the minimal length of the time interval between two consecutive reversals. Then we enlarge the window width to 50, 100 and 500 time units. In every panel is clear that when approaching the reversal the standard deviation and the variance increase. However in fig. 3 we see that these parameters are not very effective to predict in advance the eminent reversal, i.e. phase transition. The enlargement of the window width should make this parameters better in forecasting the future transitions. This can be seen in figures 4 – 6 where the standard deviation and variance become better in forecasting the eminent reversal when the window width increases.

Figure 5: Standard deviation (dashed) and variance (dotted) for the interval with width 100 time units.

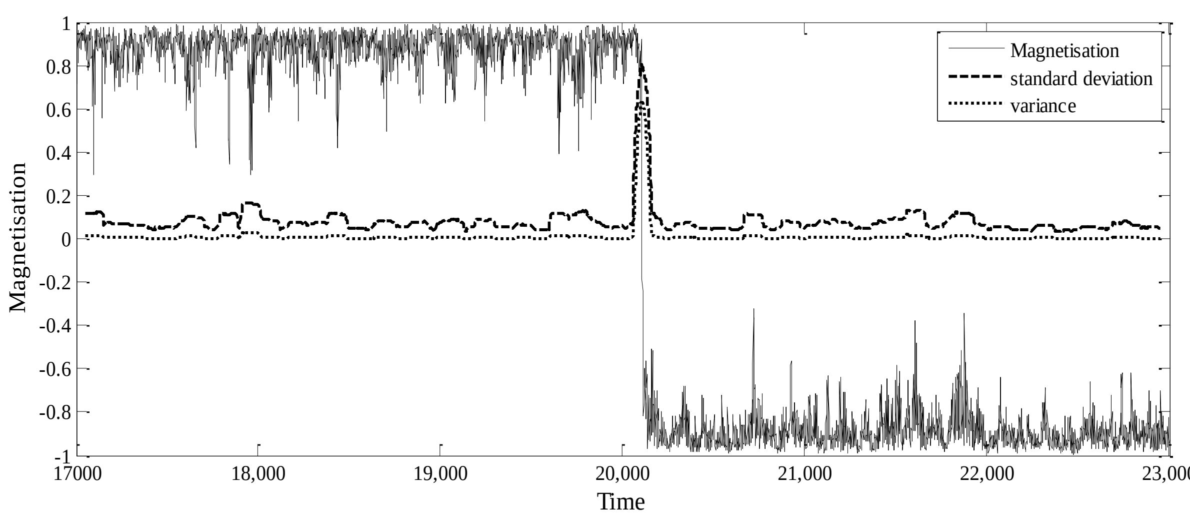
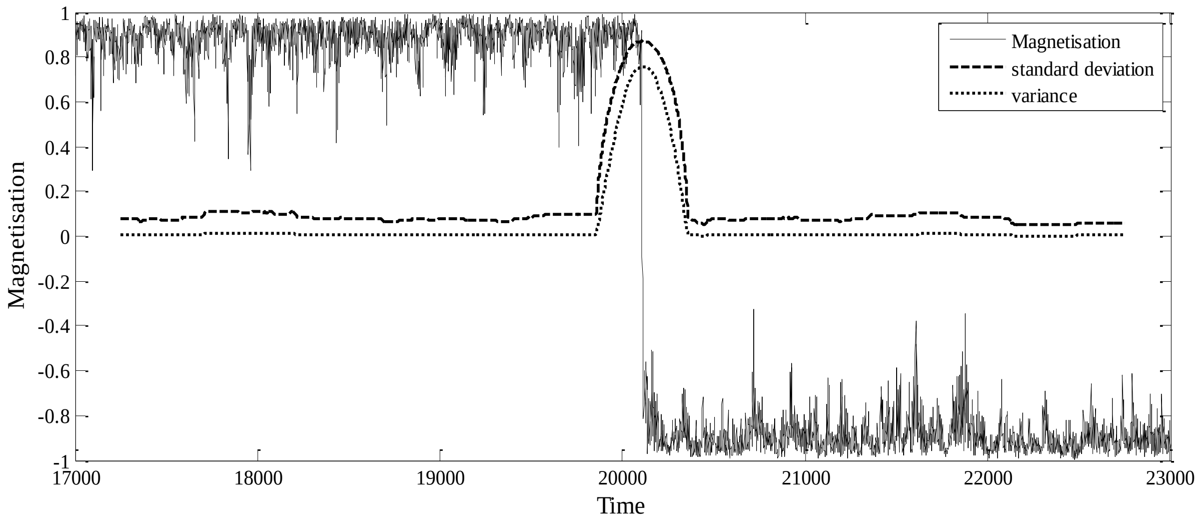


Figure 6: Standard deviation (dashed) and variance (dotted) for the interval with width 500 time units

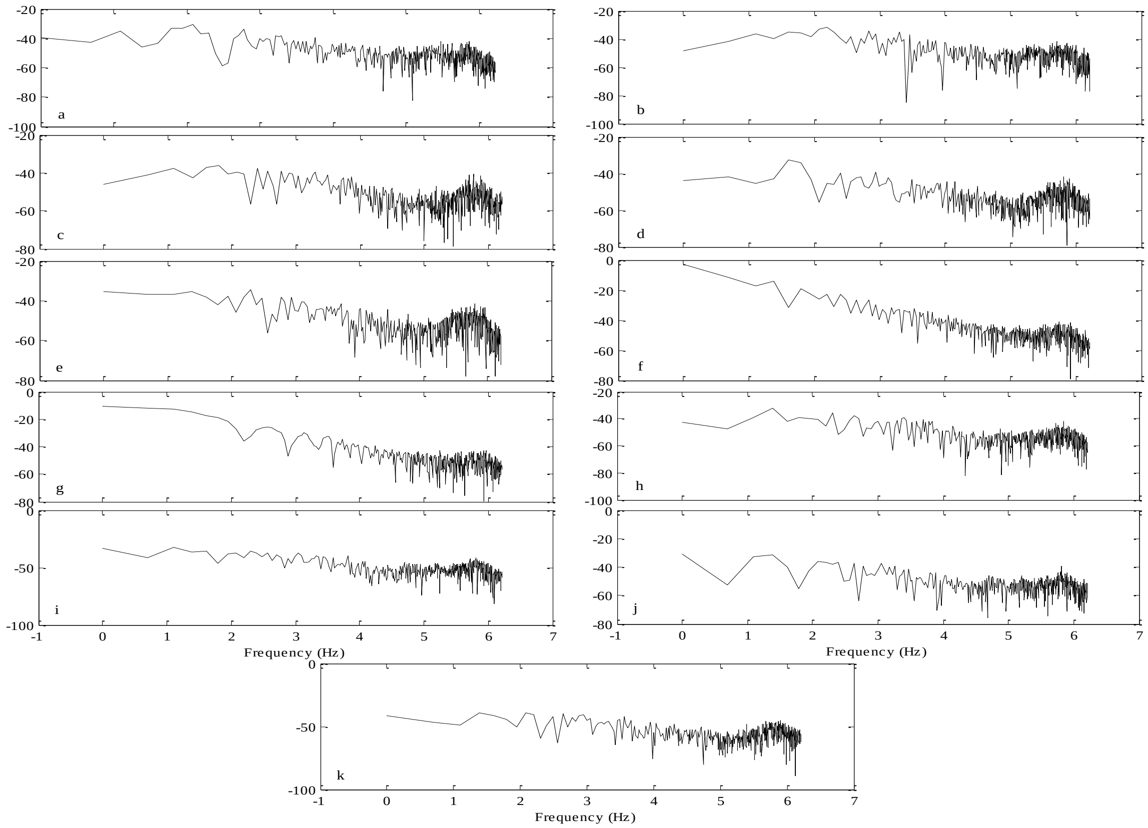


When the window width is small normally the higher frequency variations like secular variation (SV) are captured. The sharp changes in magnetisation are reflected to the sharp changes in standard deviation and variance. When the window width is increased then the smaller variations are somehow obscured. This becomes evident in fig. 6 where these parameters are almost constant except for the middle section of the time series where is located the reversal.

The reversal is the phenomenon with the largest period, i.e. smallest frequency, in the spectrum of temporal variations of the geomagnetic field. As such, the inclusion of a reversal in the time series is reflected in the increase of the weight of low frequencies in the PSD of the time series. In fig. 7 are shown 11 panels where for each panel is calculated the PSD for a time series of 1000 time units. Then the window slides to cover the whole time series. The increase in low frequencies can be seen in panel f where the reversal is included.

There can be seen that the weight of low frequencies diminishes when going further away from the reversal in both directions. However in panels e and g the weight of low frequencies in the PSDs is significantly larger than the cases of the windows at the edges of the time series (the rest of the panels). In these last cases the low frequency phenomenon of the reversal is not included. This sequence of PSDs shows that when approaching a reversal there is a reflection in the frequency contribution which signals the forthcoming phase transition.

Figure 7: Power Spectrum Density (PSD) with a window width of 1000 time units. Panels: a) 1 – 1000, b) 500 – 1500, c) 1000 – 2000, d) 1500 – 2500, e) 2000 – 3000, f) 2500 – 3500 (reversal included), g) 3000 – 4000, h) 3500 – 4500, i) 4000 – 5000, j) 4500 – 5500, k) 5000 – 6000. Frequency is in the linear scale, while the vertical axis is in log scale



Discussions and conclusion

In this paper we analysed a time series produced by a stochastic model of the geomagnetic field from the perspective of statistical analysis to find possible early warning signals or indicators that could serve as possible methods to forecast future phase transitions also known as reversal of polarity of the dipolar field.

When approaching the reversal/transition the values of magnetisation tend to become more distant to each other and the difference between consecutive values arises when approaching the transition. Therefore the values of standard deviation and variance increase, too. The maximum value of both statistical parameters is achieved when the midpoint of the reversal coincides with the midpoint of the window employed for the calculation of the parameter. Hence the maximum value of the parameters is located in the middle of the reversal interval. When leaving the reversal interval, the subsequent values of magnetisation become more like to each other and consequently the values of standard deviation and variance decrease considerably. However the window width substantially affects the ability of a given statistical parameter to potentially predict the phase transition. The optimal window width has to be chosen by trial-and-error taking into consideration that an exceedingly small width does not allow an early forecast, while on the other hand a large width obscures many details.

The approach to an eminent transition is reflected to the PSD of the time series. The reversal lies in the low frequency end of the spectrum of time variations of the geomagnetic field. The inclusion of the reversal in a time series increases the weight of low frequencies. This increase is reflected in the PSD with a substantial negative increase in the slope for the section low-to-middle frequencies. When getting away from the reversal the slope in that specific part of the PSD decreases considerably as can be seen from panels a-d and h-k in fig. 7.

In this study we have confined ourselves to reversals only. It would be very interesting to perform the analysis of standard deviation, variance and PSD when there are included excursions. This phenomenon is considered to be an aborted reversal (Valet et al., 2008). Also excursions have a higher frequency compared to reversals and their presence may be reflected in principle by an increase of the weight in the middle-to-low frequency band. However this hypothesis remains to be studied in the future.

There can be identified a problem with the methods described in this paper. In fig. 3-4 can be seen that not always an increase in standard deviation of variance leads to a transition. This fact complicates the analysis in real world time series where temporary increases of the statistical parameters in use may erroneously lead to the idea that a transition will soon occur. The application of the methods described here in real world time series that do not contain phase transitions is extremely crucial in validating or discarding the methods themselves.

It would be of much interest to test other statistical parameters for geomagnetic time series to eventually find other possible early warning signals. Possible candidates are: autocorrelation at-lag-1, detrended fluctuation analysis, skewness or kurtosis. Very useful would be the use of models like potential well estimator, to study the time series from early warning signals perspective. These tests remain to be done in forthcoming studies.

The “domino” model is considered to be a toy model and actually mimics the geomagnetic dipolar field. However the analysis of real geomagnetic time series constructed from palaeomagnetic measurements would naturally be the next study to be done. Very fruitful would be the analysis of time series generated by full dynamo models that are considered to be very close to real geomagnetic field time series. These studies may lead to a refinement of the present methods or to the invention of new methods.

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